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Elliptic and Parabolic Systems for Neutron Fission and Diffusion

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1. INTRODUCTION, NONLINEAR SYSTEMS FOR NEUTRON FISSION WITH TEMPERATURE FEEDBACK

This paper considers multigroup diffusion equations, describing neutron-flux reaction–diffusion inside a nuclear fission reactor. Both steady states and large-time dynamic behavior are studied. The reactor core is represented by a bounded domain $\mathscr{D}$ in R^d, $d \geqslant 2$. The functions $u_i(x)$ or $\tilde{u}_i(x, t)$, $i = 1,\ldots, n$, $x = (x_1,\ldots, x_d) \in \mathscr{D}$ are the neutron flux of the ith energy group (decreasing energy for increasing i). $T(x)$ is the core temperature above average coolant temperature. The following system of nonlinear temperature feedback multigroup elliptic diffusion equations will be considered in Section 2:

$$\begin{aligned} \Delta u_i + \sum_{j=1}^{n} H_{ij}(x, T)u_j = 0, \qquad & i = 1,\ldots, n, \\ \Delta T - c(x)T + \sum_{j=1}^{n} G_j(x, T)u_j = 0 \qquad & \text{in } \mathscr{D}. \end{aligned} \tag{1.1}$$

Here $\Delta \equiv \sum_{i=1}^{d} \partial^2/\partial x_i^2$, $c(x) > 0$ in $\mathscr{D}$ closure represents the cooling function. The functions determining interaction rates, $H_{ij}(x, T)$ and $G_j(x, T)$, are assumed to be functions of space and temperature. In nuclear engineering terminologies, H_{ij} describe fission, removal, group-transfer and

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absorption "cross sections," taking into account the parameters of diffusion, neutron and energy release. Details can be found in [2, p. 288].

Linear models, neglecting T and the last equation in (1.1), had been extensively studied analytically and numerically, see, e.g., [2, 4, 7, 15]. The advisability of a temperature-dependent nonlinear feedback model has been proposed and studied in [8, 9, 1, 12]. As temperature changes, materials in the core may contract, expand, or change phase, eventually causing a change in macroscopic cross section.

In [12], temperature feedback models are only investigated for two group neutron flux where the group-transfer scattering effect is quite simple. In practice, the multigroup equations are commonly applied in cases of four or more groups. In [1], more than two groups are considered; however, the conditions are too restrictive and the essense of the mathematical deduction is disguised. Our results here consider much more general scattering and fission formations of various groups of n. Moreover, the coefficients or cross sections H_{ij} are now dependent on space x, while in [12] they are independent of x and can only be applied to homogeneous reactors. Further, the mathematical relations are much more clearly presented in matrix notations in this present paper.

We now clarify our notations, conventions, and assumptions. $\mathscr{D}$ is a bounded domain in R^d, $d \geqslant 2$, whose boundary $\delta\mathscr{D}$ is C^2 smooth (i.e., can be locally represented as $x_i = \phi(x)$ for some i, ϕ with continuous second derivatives and independent of x_i); $\bar{\mathscr{D}}$ denotes $\mathscr{D}$ closure. The functions H_{ij}, G_i $i, j = 1,..., n$, are continuous functions of $x \in \mathscr{D}$, $T \geqslant 0$; $c(x)$ is continuous and positive in $\bar{\mathscr{D}}$. For convenience, let

$$\begin{aligned} \tilde{h}_{ij} &= \inf\{H_{ij}(x, T) \mid x \in \bar{\mathscr{D}}, T \geqslant 0\}, \\ \bar{h}_{ij} &= \sup\{H_{ij}(x, T) \mid x \in \mathscr{D}, T \geqslant 0\} \end{aligned} \tag{1.2}$$

for $i, j = 1,..., n$. Similarly, define $\tilde{g}_i$, $\bar{g}_i$ to be the corresponding inf and sup of G_i, $i = 1,..., n$. We will always assume that

$$-\infty < \tilde{h}_{ij} \leqslant \bar{h}_{ij} < \infty, \qquad 0 \leqslant \tilde{g}_i \leqslant \bar{g}_i < \infty \tag{C1}$$

for $i, j = 1,..., n$. For $i \neq j$, H_{ij} describe group transfer and fissions of neutrons from other groups; while H_{ii} is affected by control rods and absorptions. Consequently, we always assume that

$$\begin{aligned} 0 \leqslant \tilde{h}_{ij} \leqslant \bar{h}_{ij} < \infty \qquad & \text{each } i, j = 1,..., n, \text{ with } i \neq j, \\ -\infty < \tilde{h}_{ii} < \bar{h}_{ii} < \infty, \qquad & i = 1,..., n. \end{aligned} \tag{C2}$$

Conditions (C1) and (C2) are very reasonable and general assumptions for the reactor model.

In some cases, we assume that an energy group i receives transfer of neutrons from some group j, $j<i$ (cf. condition (II) in Theorem 2.1). In another case, a related "irreducible" condition is assumed (cf. condition (II*) in Theorem 2.3). For later conveniences, we define:

$$\tilde{H} \text{ and } \bar{H} \text{ to be } n\times n \text{ square matrices whose } (i,j)\text{th entries are, respectively, } \tilde{h}_{ij} \text{ and } \bar{h}_{ij} \text{ for } 1\leqslant i,j\leqslant n. \tag{1.3}$$

Let $\lambda_1>0$ denote the first eigenvalue of the eigenvalue problem: $\Delta w+\lambda w=0$ in $\mathscr{D}$, $w=0$ in $\delta\mathscr{D}$, where $w=\omega(x)$ is the corresponding normalized eigenfunction with $\max\{\omega(x)\mid x\in\bar{\mathscr{D}}\}=1$. For positive integers r, $C^r(\mathscr{D})$ and $C^r(\bar{\mathscr{D}})$ denote r-times continuously differentiable functions in $\mathscr{D}$ and $\bar{\mathscr{D}}$, respectively.

In Section 2, we consider Eqs. (1.1) in $\mathscr{D}$ with nonnegative or zero Dirichlet boundary conditions on $\delta\mathscr{D}$. Theorem 2.1 gives some very readily applicable criteria when certain components of nonnegative steady-state solutions must be identically zero. It can thus be considered as a reduction theorem, with whose application it suffices to investigate a proper subsystem of (1.1). Theorem 2.2 considers the corresponding analog for the time-dependent parabolic system; when certain components has positive initial conditions, they would "blow up" as $t\to+\infty$. Theorem 2.3 is a variant of Theorem 2.1 with a somewhat stronger hypothesis, under which the Perron–Frobenius theory on positive eigenvectors for nonnegative matrices can be applied. Theorem 2.4 considers a condition complementary to that of (I) in Theorem 2.1. It finds a "decay" criterion (2.12) for the nonexistence of nontrivial nonnegative solutions; such condition for the corresponding time-dependent parabolic problem would imply solutions tending to zero as $t\to+\infty$. Corollary 2.5 gives a simple diagonally dominant criterion for an application of Theorem 2.4.

In Section 3, a simplier version of temperature feedback is considered. We assume that fissions and other cross sections are promptly affected by the neutron-flux, eliminating the last equation in Eqs. (1.1). Theorem 3.1 finds sufficient conditions for the existence of nontrivial nonnegative equilibria. Conditions (3.3) and (3.4) enable the success in constructing positive lower and upper solutions, respectively, for the problem (3.1), (3.2).

2. Criteria for Blow-Up and Decay, Connection with Perron–Frobenius Eigenvectors

In this section, hypotheses (C1) and (C2) are assumed in every theorem and corollary. Theorem 2.1 gives some very readily applicable cirteria when certain component of a nonnegative steady-state solution must be identically zero. A "sweeping" argument of Serrin's is used in the proof.

THEOREM 2.1. *Suppose that the* $n \times n$ *matrix* $\tilde{H} - \lambda_1 I$ *has a square* $m \times m$ *submatrix* $A = (a_{ij})$, *(formed by deleting the* k_1th,..., k_{n-m}th *rows and columns of* $\tilde{H} - \lambda_1 I$, $1 \leqslant k_1 < k_2,..., < k_{n-m} \leqslant n$, $1 \leqslant m \leqslant n$*) with the properties that:*

(I) $A\mathbf{c} > 0$ *for some positive* m *vector* $\mathbf{c} > 0$*; and*

(II) *For each* $i = 2,..., m$ *at least one of* $a_{i1}, a_{i2},..., a_{ii-1}$ *is positive.*

Then Eq. (1.1) *has no solution* $(\hat{u}_1(x),..., \hat{u}_n(x), \hat{T}(x))$ *with all the following three properties:*

(i) *each component is in* $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$,

(ii) $\hat{u}_i(x) \geqslant 0$, $i = 1,..., n$, $\hat{T}(x) \geqslant 0$ *in* $\bar{\mathscr{D}}$,

(iii) $\hat{u}_{s_1}(x) \not\equiv 0$ *in* $\bar{\mathscr{D}}$, *where* s_1 *is the first positive integer not included in* $k_1,..., k_{n-m}$.

(Roughly speaking, any solution of (1.1) satisfying (i) and (ii), must have its s_1th component identically zero. When $m = n$,..., no k_i row or column is deleted, and $\tilde{H} - \lambda_1 I \equiv A$. A vector $\mathbf{v} > 0$ means that each component of $\mathbf{v}$ is positive.)

Proof. Let $1 \leqslant s_1 < s_2 \cdots < s_m \leqslant n$ be integers not in $\{k_1,..., k_{n-m}\}$, and thus $a_{ij} = \tilde{h}_{s_i s_j} - \lambda_1 \delta_{ij}$, $1 \leqslant i, j \leqslant m$, where δ_{ij} is the Kronecker delta. Assume that $(\hat{u}_1(x),..., \hat{u}_n(x), \hat{T}(x))$ exists as described with properties (i), (ii), and (iii). We will construct a family of lower bounds for the functions $\hat{u}_{s_i}(x)$, $i = 1,..., m$, parametrized by $\delta > 0$. As $\delta \to \infty$, the lower bound will tend to ∞. For each $\delta > 0$, define $u^\delta_{s_i}(x) = c_i \delta\omega(x)$ for $x \in \bar{\mathscr{D}}$, $i = 1,..., m$, where $\operatorname{col}(c_1,..., c_n) = \mathbf{c}$ is the vector described in (I) above; defined $u^\delta_{k_i}(x) \equiv 0$, $i = 1,..., n-m$, and $T^\delta(x) \equiv 0$. For each $i = 1,..., m$, when $u_j(x) \geqslant u^\delta_j(x)$ for all $j \neq s_i$, and $T(x) \geqslant T^\delta(x)$, we have

$$\begin{aligned}
&\Delta u^\delta_{s_i}(x) + H_{s_i s_i}(x, T(x))\, u^\delta_{s_i}(x) + \sum_{j=1, j \neq s_i}^{n} H_{s_i j}(x, T(x))\, u_j(x) \\
&\quad \geqslant [-\lambda_1 + \tilde{h}_{s_i s_i}]\, c_i\, \delta\omega(x) + \sum_{j=1, j \neq i}^{m} \tilde{H}_{s_i s_j} u^\delta_{s_j} \\
&\quad = [-\lambda_1 + \tilde{h}_{s_i s_i}]\, c_i\, \delta\omega + \sum_{j=1, j \neq i}^{m} a_{ij} c_j\, \delta\omega = (A\mathbf{c})_i\, \delta\omega(x) > 0. \qquad (2.1)
\end{aligned}$$

We now show that properties (i)–(iii) imply that $\hat{u}_{s_i}(x) > 0$ for $x \in \mathscr{D}$, $i = 1,..., m$. Let $C > \max\{|\tilde{h}_{s_1 s_1}|, |\bar{h}_{s_1 s_1}|\}$, we have

$$\Delta\hat{u}_{s_1}(x) - C\hat{u}_{s_1}(x)$$
$$= -[H_{s_1s_1}(x, \hat{T}(x)) + C]\,\hat{u}_{s_1}(x) - \sum_{j=1, j\neq s_1}^{n} H_{s_1j}(x, \hat{T}(x))\,\hat{u}_j(x) \leqslant 0$$

in $\mathscr{D}$, $\hat{u}_{s_1} \geqslant 0$ in $\bar{\mathscr{D}}$. The maximum principle implies that $\hat{u}_{s_1}(x) > 0$ in $\mathscr{D}$. Similarly, considering $\Delta\hat{u}_{s_i} - P\hat{u}_{s_i} \leqslant 0$ in $\mathscr{D}$ for large enough $P > 0$, we deduce from the maximum principle that $\hat{u}_{s_i} > 0$ in $\mathscr{D}$ or $\hat{u}_{s_i} \equiv 0$ in $\bar{\mathscr{D}}$, $i = 2, ..., m$. However, property (II) implies successively that the trivial function is not a solution of the s_ith equation in (1.1), $i = 2, ..., m$. Hence $\hat{u}_{s_i}(x) > 0$ for $x \in \mathscr{D}$. Moreover, the maximum principle at the boundary indicates that the outward normal derivatives $\partial\hat{u}_{s_i}/\partial\eta$ are negative at those boundary points where the corresponding function is 0.

From the above paragraph, we see that the set

$$\mathscr{J} \equiv \{\tau > 0 \mid \hat{u}_{s_i}(x) > u^\delta_{s_i}(x),\ i = 1, ..., m, \text{ for all } 0 \leqslant \delta < \tau,\ x \in \mathscr{D}\}$$

is nonempty. Suppose $\mathscr{J}$ has an upper bound, let its lub be $\bar{\delta}$. If there is a point at the boundary where $u^{\bar{\delta}}_{s_i} = \hat{u}_{s_i}$, some $i = 1, ..., m$, we deduce a contradiction to the definition of $\bar{\delta}$ by using the maximum principle at the boundary, with the inequality:

$$\begin{aligned} &\Delta(\hat{u}_{s_i} - u^{\bar{\delta}}_{s_i}) - P(\hat{u}_{s_i} - u^{\bar{\delta}}_{s_i}) \\ &\quad = \left\{\Delta\hat{u}_{s_i} + \sum_{j=1}^{n} H_{s_ij}(x, \hat{T})\,\hat{u}_j\right\} \\ &\quad\quad - \left\{\Delta u^{\bar{\delta}}_{s_i} + H_{s_is_i}(x, \hat{T})\,u^{\bar{\delta}}_{s_i} + \sum_{j=1, j\neq s_i}^{n} H_{s_ij}(x, \hat{T})\hat{u}_j\right\} \\ &\quad\quad - \{H_{s_is_i}(x, \hat{T}) + P\}(\hat{u}_{s_i} - u^{\bar{\delta}}_{s_i}) \leqslant 0 \end{aligned} \tag{2.2}$$

in $\mathscr{D}$. (The last inequality is true for $P > \max\{|\tilde{h}_{s_is_i}|, |\bar{h}_{s_is_i}|\}$, and due to inequality (2.1), for $\hat{u}_j(x) \geqslant u^{\bar{\delta}}_j(x)$, $j \neq s_i$, $\hat{T} \geqslant T^{\bar{\delta}}$.) Contradiction arises because (2.2) implies that $\partial\hat{u}_{s_i}/\partial\eta < \partial u^{\bar{\delta}}_{s_i}/\partial\eta$ at those points at the boundary where $\hat{u}_{s_i} = u^{\bar{\delta}}_{s_i}$, and thus $u^{\bar{\delta}+\varepsilon}_{s_i} < \hat{u}_{s_i}$ for all $x \in \mathscr{D}$, some small $\varepsilon > 0$. On the other hand, suppose that there is a point $\bar{x} \in \mathscr{D}$, where $u^{\bar{\delta}}_{s_i}(\bar{x}) = \hat{u}_{s_i}(\bar{x})$, some $i = 1, ..., m$. Inequality (2.2) and the maximum principle imply that $u^{\bar{\delta}}_{s_i}(x) \equiv \hat{u}_{s_i}(x)$ in $\bar{\mathscr{D}}$. However, we consider in $\mathscr{D}$:

$$\begin{aligned} 0 &= \Delta\hat{u}_{s_i} + H_{s_is_i}(x, \hat{T})\,\hat{u}_{s_i} + \sum_{j=1, j\neq s_i}^{n} H_{s_ij}(x, \hat{T})\hat{u}_j \\ &= \Delta u^{\bar{\delta}}_{s_i} + H_{s_is_i}(x, \hat{T})\,u^{\bar{\delta}}_{s_i} + \sum_{j=1, j\neq s_i}^{n} H_{s_ij}(x, \hat{T})\,\hat{u}_j > 0, \end{aligned} \tag{2.3}$$

which is a contradiction. The last inequality is true by letting $\delta = \bar{\delta}$ in (2.1).

The last paragraph shows that the set $\mathscr{J}$ is unbounded. However, as $\delta \to +\infty$, $u_{s_i}^{\delta}(x) \to +\infty$ for $x \in \mathscr{D}$, $i = 1,..., m$. This proves the nonexistence of $(\hat{u}_1(x),..., \hat{u}_n(x), \hat{T}(x))$.

Essentially Theorem 2.1 asserts that under assumptions (I), (II) and (i), (ii), the s_1th component must be identically zero; otherwise it cannot be finite. The corresponding analog in the parabolic case asserts further that all the s_1th,..., s_mth components must tend to $+\infty$ as $t \to +\infty$. This is the context of the following Theorem 2.2.

THEOREM 2.2. *Suppose that the $n \times n$ matrix $\tilde{H} - \lambda_1 I$ has a square $m \times m$ submatrix $A = (a_{ij})$ (formed as described in Theorem 2.1), with the property that*

(I) *$A\mathbf{c} > 0$ for some positive m vector $\mathbf{c} > 0$.*

Let $(\tilde{u}_1(x, t),..., \tilde{u}_n(x, t), \tilde{T}(x, t))$ be a solution of

$$\begin{aligned} \frac{\partial \tilde{u}_i}{\partial t} &= \Delta \tilde{u}_i + \sum_{j=1}^{n} H_{ij}(x, \tilde{T})\tilde{u}_j, \qquad i = 1,..., n, \\ \frac{\partial \tilde{T}}{\partial t} &= \Delta \tilde{T} - c(x)\tilde{T} + \sum_{j=1}^{n} G_j(x, \tilde{T})\tilde{u}_j \end{aligned} \tag{2.4}$$

for $(x, t) \in \mathscr{D} \times (0, \infty)$, with each component function in $C^2(\mathscr{D} \times (0, \infty)) \cap C^1(\bar{\mathscr{D}} \times [0, \infty))$ and initial-boundary conditions satisfying the conditions

$$\begin{aligned} &\tilde{u}_i(x, 0) > 0 \qquad \text{for} \quad x \in \mathscr{D}, \\ &\frac{\partial \tilde{u}_i}{\partial \eta}(x, 0) < 0 \qquad \text{for} \quad x \in \delta\mathscr{D}, i = s_1,..., s_m \end{aligned} \tag{2.5}$$

(recall that $s_1,..., s_m$ are those integers between 1 to n not included in $\{k_1,..., k_{n-m}\}$ described in Theorem 2.1), and

$$\begin{aligned} &\tilde{u}_i(x, t) \geqslant 0, \quad i = 1,..., n, \\ &\tilde{T}(x, t) \geqslant 0 \quad \text{for } (x, t) \in (\bar{\mathscr{D}} \times \{0\}) \cup (\delta\mathscr{D} \times (0, \infty)). \end{aligned} \tag{2.6}$$

Then $\tilde{u}_i(x, t) \to +\infty$ for all $x \in \mathscr{D}$, $i = s_1,..., s_m$ as $t \to +\infty$. (More precisely, for such x and i, we have $\tilde{u}_i(x, t) \geqslant \varepsilon_1 \omega(x) e^{\varepsilon_2 t}$ for some positive constants ε_1, ε_2 and all $t \in [0, \infty)$.)

Proof. For convenience, let $S = \{s_1,..., s_m\}$. For $1 \leqslant i \leqslant n+1$, $i \notin S$, define $v_i \equiv 0$. In (I), denote $\mathbf{c} = \mathrm{col}(c_1,..., c_m)$. Choose ε so that

$0<\varepsilon<\min\{(A\mathbf{c})_i \mid i=1,\ldots,m\}$; and choose $\delta>0$ so that $\delta c_i\omega(x)<\tilde{u}_i(x,0)$ for all $x\in\mathscr{D}$, $i\in S$. Define for

$$i\in S:\quad v_i(x,t)=\delta c_i\omega(x)\,e^{\varepsilon t}\qquad \text{for}\quad (x,t)\in\bar{\mathscr{D}}\times[0,\infty).$$

Consider the set:

$$\mathscr{B}\equiv\{(x,t,z_1,\ldots,z_{n+1})\mid(x,t)\in\mathscr{D}\times(0,\infty),\ z_i\geqslant v_i(x,t),\ i=1,\ldots,n+1\}$$

Clearly, we have for each $i\in S$, $(x,t,z_1,\ldots,z_{n+1})\in\mathscr{B}$:

$$\begin{aligned}
&\Delta v_i+H_{ii}(x,z_{n+1})v_i+\sum_{j=1,j\neq i}^{n}H_{ij}(x,z_{n+1})z_j-\frac{\partial v_i}{\partial t}\\
&\quad\geqslant[-\lambda_1+\tilde{h}_{ii}]\,\delta c_i\omega(x)\,e^{\varepsilon t}\\
&\qquad+\sum_{j\in S,j\neq i}H_{ij}(x,z_{n+1})\,\delta c_j\omega(x)\,e^{\varepsilon t}-\varepsilon\,\delta c_i\omega(x)\,e^{\varepsilon t}\\
&\quad\geqslant[-\lambda_1+\tilde{h}_{ii}-\varepsilon]\,\delta c_i\omega e^{\varepsilon t}+\sum_{j\in S,j\neq i}\tilde{h}_{ij}\,\delta c_j\omega e^{\varepsilon t}\\
&\quad=\{(A\mathbf{c})_k-\varepsilon\}\,\delta\omega(x)\,e^{\varepsilon t}>0
\end{aligned}\tag{2.7}$$

(here k is the integer where $s_k=i$). For $1\leqslant i\leqslant n$, $i\notin S$, we have for $(x,t,z_1,\ldots,z_{n+1})\in\mathscr{B}$:

$$\Delta v_i+H_{ii}(x,z_{n+1})v_i+\sum_{j=1,j\neq i}^{n}H_{ij}(x,z_{n+1})z_j-\frac{\partial v_i}{\partial t}\geqslant\sum_{j=1,j\neq i}^{n}\tilde{h}_{ij}z_j\geqslant 0.\tag{2.8}$$

Finally, for $(x,t,z_1,\ldots,z_{n+1})\in\mathscr{B}$, we have

$$\Delta v_{n+1}-c(x)v_{n+1}+\sum_{j=1}^{n}G_j(x,v_{n+1})z_j-\frac{\partial v_{n+1}}{\partial t}\geqslant\sum_{j=1}^{n}\tilde{g}_jz_j\geqslant 0.\tag{2.9}$$

Moreover, at $t=0$ and for $x\in\delta\mathscr{D}$, v_i's satisfy

$$v_i(x,t)\leqslant\tilde{u}_i(x,t),\quad i=1,\ldots,n,\qquad v_{n+1}(x,t)\leqslant\tilde{T}(x,t)\tag{2.10}$$

for $(x,t)\in(\bar{\mathscr{D}}\times\{0\}\cup(\delta\mathscr{D}\times(0,\infty))$. From inequalities (2.7) to (2.10) we conclude (see e.g. Theorem 1 in [3]) that (2.10) is true for $(x,t)\in\bar{\mathscr{D}}\times[0,\infty)$. Consequently, for $i=s_1,\ldots,s_m$, we have

$$\tilde{u}_i(x,t)\geqslant\delta c_i\omega(x)\,e^{\varepsilon t}\qquad\text{for}\quad(x,t)\in\bar{\mathscr{D}}\times[0,\infty).$$

Remark. In Theorem 2.2, suppose that (2.4) is modified by changing $\Delta\tilde{u}_i$ to $\sigma_i\Delta\tilde{u}_i$ for each $i=1,\ldots,n$, with $\sigma_i>0$, and $\Delta\tilde{T}$ to $\sigma\,\Delta\tilde{T}$, with $\sigma>0$.

Theorem 2.2 is true verbatim except with $\tilde{H}-\lambda_i I$ changed to $\tilde{H}-\operatorname{diag}(\sigma_1\lambda_1, \sigma_2\lambda_1, \ldots, \sigma_n\lambda_1)$.

Suppose that in Theorem 2.1, hypothesis (II) is modfied to a more restrictive irreducible assumption, we can prove that more components of classical nonnegative solutions of (1.1) must be identically zero. Hypothesis (II) in Theorem 2.1 does not make any assumption on the entries above the diagonal of A, and one can thus readily find a reducible matrix A satisfying hypothesis (II). Theorem 2.3 below is therefore a somewhat more restrictive version of Theorem 2.1.

THEOREM 2.3. *Suppose that the* $n \times n$ *matrix* $\tilde{H}-\lambda_1 I$ *has a square submatrix* $A=(a_{ij})$ (*formed as described in Theorem* 2.1), *with the properties that*:

(I) $A\mathbf{c}>0$ *for some positive* m *vector* $\mathbf{c}>0$; *and*

(II*) A *is irreducible.*

Then, any solution $(\hat{u}_1(x), \ldots, \hat{u}_n(x), \hat{T}(x))$ *of Eq.* (1.1) *with the properties that*:

(i) *each component is in* $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$, *and*

(ii) $\hat{u}_i(x) \geqslant 0$, $i=1, \ldots, n$, $\hat{T}(x) \geqslant 0$ *in* $\bar{\mathscr{D}}$, *must satisfy*:

$$\hat{u}_{s_i}(x) \equiv 0 \qquad \text{in } \bar{\mathscr{D}} \qquad \text{for} \quad i=1, \ldots, m.$$

(Recall that s_i's are those integers between 1 and n not in $\{k_1, \ldots, k_{n-m}\}$ described in Theorem 2.1.)

Proof. Since the off-diagonal entries of A are all nonnegative, property (II*) implies by means of the Perron–Frobenius theory (see, e.g., [6]) that there is a positive m row vector $\mathbf{e}'$ so that $\mathbf{e}'A=r\mathbf{e}'$ for a real number r. We have

$$r(\mathbf{e}' \cdot \mathbf{c})=(\mathbf{e}'A) \cdot \mathbf{c}=\mathbf{e}' \cdot (A\mathbf{c})>0$$

by property (I). Consequently $r>0$, because $\mathbf{e}'$ and $\mathbf{c}$ are positive. Let $z(x)=\sum_{i=1}^m e_i \hat{u}_{s_i}(x)$, where $\mathbf{e}'=(e_1, \ldots, e_m)$. We have for $x \in \mathscr{D}$:

$$\begin{aligned}
\Delta z+\lambda_1 z &= \sum_{i=1}^m e_i \sum_{j=1}^n -H_{s_i j}(x, \hat{T}(x))\, \hat{u}_j+\sum_{i=1}^m e_i \lambda_1 \hat{u}_{s_i} \\
&\leqslant -\sum_{i=1}^m e_i([\tilde{H}-\lambda_1 I]\hat{u})_{s_i} \\
&\leqslant -\sum_{i=1}^m e_i(A\hat{\hat{u}})_i=-r\mathbf{e}' \cdot \hat{\hat{u}}=-rz
\end{aligned} \tag{2.11}$$

(here $\hat{u} =^{(\mathrm{def})} \mathrm{col}(\hat{u}_1(x),..., \hat{u}_n(x))$, $\hat{u} =^{(\mathrm{def})} \mathrm{col}(\hat{u}_{s_1}(x),..., \hat{u}_{s_m}(x))$.) From (2.11) we have

$$\begin{aligned}
0 &\geqslant \int_{\mathscr{D}} \omega(x)\{(\Delta + \lambda_1 + r)z\}\, dx \\
&= \int_{\delta\mathscr{D}} (-z) \frac{\partial \omega}{\partial \eta}\, d\sigma + \int_{\mathscr{D}} z\, \Delta\omega\, dx + \int_{\mathscr{D}} (\lambda_1 + r)\, \omega z\, dx \\
&\geqslant \int_{\mathscr{D}} -\lambda_1 z\omega\, dx + \int_{\mathscr{D}} (\lambda_1 + r)\, \omega z\, dx \\
&= \int_{\mathscr{D}} rz\omega\, dx \geqslant 0.
\end{aligned}$$

In order not to have a contradiction above, we must have $z(x) \equiv 0$ in $\bar{\mathscr{D}}$. Consequently, we have $\hat{u}_{s_i} \equiv 0$ in $\bar{\mathscr{D}}$ for $i = s_1,..., s_m$.

The remaining part of this section dicusses a condition when neutron density of each group will decay to zero for the time-dependent parabolic model. This means that the only nonnegative steady state is the trivial one. Condition (2.12) in Theorem 2.4 is nearly the reverse of condition (I) in Theorem 2.1.

THEOREM 2.4. *Suppose that the $n \times n$ matrix $\bar{H} - \lambda_1 I$ has the property that*:

$$(\bar{H} - \lambda_1 I)\mathbf{y} < 0 \text{ for some positive } n \text{ vector } \mathbf{y} > 0. \tag{2.12}$$

Then Eq. (1.1) *with boundary conditions*

$$u_i(x) = 0, \quad i = 1,..., n, \qquad T(x) = 0 \quad \text{for } x \in \delta\mathscr{D}$$

has the solution $(0,..., 0)$ *as the only solution with the properties that each component is in* $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$ *and nonnegative in* $\bar{\mathscr{D}}$.

(Note that (2.12) implies that all the diagonal entries of $\bar{H} - \lambda_1 I$ are negative, because all its off-diagonal entries are nonnegative).

Proof. We will consider the related parabolic system (2.4) for $(x, t) \in \mathscr{D} \times (0, \infty)$, with boundary conditions

$$\tilde{u}_i(x, t) = 0, \quad i = 1,..., n, \qquad \tilde{T}(x, t) = 0 \tag{2.13}$$

for $(x, t) \in \delta\mathscr{D} \times (0, \infty)$. Here $\tilde{u}_i$, $i = 1,..., n$, $\tilde{T}$ are functions in $\bar{\mathscr{D}} \times [0, \infty)$. We will prove that all solutions of (2.4), (2.13) with components in $C^2(\mathscr{D} \times (0, \infty)) \cap C^1(\bar{\mathscr{D}} \times [0, \infty))$ and initial conditions which are nonnegative for all $x \in \bar{\mathscr{D}}$, $t=0$, will tend to zero as $t \to +\infty$. Consequently, the equilibrium solution as stated in the theorem can only be the trivial one.

Define $v_1 \equiv v_2 \equiv \cdots \equiv v_{n+1} \equiv 0$. Let $k>0$ be a constant such that $kc_i\omega(x) \geqslant \tilde{u}_i(x,0)$, $i=1,\ldots,n$, for each $x \in \bar{\mathscr{D}}$ (here c_i is the ith component of the vector $\mathbf{y}$ stated in the theorem). Let $d=\min\{c(x) \mid x \in \bar{\mathscr{D}}\}$ and σ a small enough constant with $0<\sigma<d$ so that inequalities (2.12) is valid with $\bar{H}-\lambda_1 I$ replaced by $\bar{H}-(\lambda_1-\sigma)I$ and $\mathbf{y}$ unchanged.

Choose $c_{n+1}>0$ so that $c_{n+1}>\max\{\max_{x\in\mathscr{D}} \tilde{T}(x,0), (d-\sigma)^{-1} \cdot \sum_{i=1}^{n} \bar{g}_i k c_i\}$. Finally, define $w_i = kc_i\omega(x)\, e^{-\sigma t}$, $i=1,\ldots,n$, and $w_{n+1} = c_{n+1}\, e^{-\sigma t}$. Consider the set

$$J \equiv \{(x, t, z_1, \ldots, z_{n+1}) \mid (x,t) \in \mathscr{D} \times (0,\infty), v_i(x,t) \leqslant z_i \leqslant w_i(x,t), \text{ each } i=1,\ldots,n+1\}.$$

Clearly, we have for each $i=1,\ldots,n$,

$$\Delta v_i + \sum_{\substack{j=1\\ j\neq i}}^{n} H_{ij}(x, z_{n+1})\, z_j + H_{ii}(x, z_{n+1}) v_i - \frac{\partial v_i}{\partial t} = \sum_{\substack{j=1\\ j\neq i}}^{n} H_{ij}(x, z_{n+1}) z_j \geqslant 0, \tag{2.14}$$

$$\Delta v_{n+1} - c(x) v_{n+1} + \sum_{j=1}^{n} G_j(x, z_{n+1}) z_j - \frac{\partial v_{n+1}}{\partial t} = \sum_{j=1}^{n} G_j(x, z_{n+1}) z_j \geqslant 0 \tag{2.15}$$

for all $(x, t, z_1, \ldots, z_{n+1}) \in J$. On the other hand, for all $(x, t, z_1, \ldots, z_{n+1}) \in J$,

$$\begin{aligned} &\Delta w_i + \sum_{\substack{j=1\\ j\neq i}}^{n} H_{ij}(x, z_{n+1}) z_j + H_{ii}(x, z_{n+1}) w_i - \frac{\partial w_i}{\partial t} \\ &\leqslant k\omega(x)\, e^{-\sigma t} \left\{ \sum_{\substack{j=1\\ j\neq i}}^{n} c_j \bar{h}_{ij} + (-\lambda_1 + \bar{h}_{ii} + \sigma) c_i \right\} < 0, \qquad i=1,2,\ldots,n, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\Delta w_{n+1} - c(x) w_{n+1} + \sum_{j=1}^{n} G_j(x, z_{n+1}) z_j - \frac{\partial w_{n+1}}{\partial t} \\ &\leqslant e^{-\sigma t} \left\{ (-d+\sigma) c_{n+1} + \sum_{j=1}^{n} \bar{g}_j k c_j \max_{x\in\bar{\mathscr{D}}} \omega(x) \right\} < 0, \end{aligned} \tag{2.17}$$

because of the choise of c_j, $j=1,\ldots, n+1$ and σ. Moreover, we have for $x \in \bar{\mathscr{D}}$,

$$\begin{aligned} v_i(x,0) \leqslant \tilde{u}_i(x,0) \leqslant w_i(x,0), &\qquad i=1,\ldots,n, \\ v_{n+1}(x,0) \leqslant \tilde{T}(x,0) \leqslant w_{n+1}(x,0); & \end{aligned} \tag{2.18}$$

and for $(x,t) \in \delta\mathscr{D} \times [0,\infty)$,

$$\begin{aligned} v_i(x,t) \leqslant \tilde{u}_i(x,t) \leqslant w_i(x,t), &\qquad i=1,\ldots,n, \\ v_{n+1}(x,t) \leqslant \tilde{T}(x,t) \leqslant w_{n+1}(x,t). & \end{aligned} \tag{2.19}$$

Therefore if such a solution $(\tilde{u}_1(x,t),\ldots,\tilde{u}_n(x,t), \tilde{T}(x,t))$ exists in $\bar{\mathscr{D}} \times [0,\infty)$, it will satisfy (2.19) for all $(x,t) \in \bar{\mathscr{D}} \times [0,\infty)$, by inequalities (2.14) to (2.19) above. (See, e.g., Lemma 2.1 in [11] or [10] for a variant of the comparison principles used here.)

Let $(\hat{u}_1(x),\ldots,\hat{u}_n(x),\hat{T}(x))$ be a solution of the boundary value problem described in the statement of the theorem, with properties as stated. It will be a solution of (2.4), (2.13) with the appropriate smoothness and nonnegative condition at $t=0$. Letting

$$\begin{aligned} &(\tilde{u}_1(x,t),\ldots,\tilde{u}_n(x,t),\tilde{T}(x,t)) \\ &\quad = (\hat{u}_1(x),\ldots,\hat{u}_n(x),\hat{T}(x)) \qquad \text{for} \quad (x,t) \in \bar{\mathscr{D}} \times [0,\infty). \end{aligned}$$

Inequality (2.19) for $(x,t) \in \bar{\mathscr{D}} \times [0,\infty)$ implies that

$$0 \leqslant \hat{u}_i(x) \leqslant k c_i \omega(x)\, e^{-\sigma t}, \quad i=1,\ldots,n, \qquad 0 \leqslant \hat{T}(x) \leqslant c_{n+1} e^{-\sigma t}$$

for $(x,t) \in \bar{\mathscr{D}} \times [0,\infty)$. Consequently, $(\hat{u}_1,\ldots,\hat{u}_n,\hat{T}) \equiv (0,\ldots,0)$.

We now observe a very direct consequence of Theorem 2.4.

COROLLARY 2.5. *Suppose that the $n \times n$ matrix $\bar{H} - \lambda_1 I$ has the properties that all its diagonal entries are negative, and it is diagonally dominant (i.e., $|\bar{h}_{ii} - \lambda_1| > \sum_{j=1, j\neq i}^{n} \bar{h}_{ij}$, $i=1,\ldots,n$). Then the boundary value problem in Theorem 2.4 has the solution $(0,\ldots,0)$ as the only solution with the properties that each comonent is in $C^2(\mathscr{D}) \cap C^1(\bar{\mathscr{D}})$ and nonnegative in $\bar{\mathscr{D}}$.*

Proof. Choose $\mathbf{y} = \mathrm{col}(1,1,\ldots,1)$ to satisfy hypothesis (2.12), and apply Theorem 2.4.

Remark. If $(\bar{H} - \lambda_1 I)\mathbf{y} = r\mathbf{y}$ for some $\mathbf{y} > 0$, $r < 0$, then we can clearly apply Theorem 2.4.

3. Positive Equilibria for Prompt Feedback Elliptic Equations

Conditions for the existence of positive steady states had not been found for the general model (1.1) in the last section, under zero Dirichlet boundary condition. To make the analysis more tractable, we consider a slightly simplier model. We now assume that the reaction coefficients (i.e., cross sections) are functions of the neutron fluxes u_i directly. That is, the feedback is prompt, and does not have to be regulated through the change in T indirectly through the last equation in (1.1). More precisely, we have

$$\Delta u_i + \sum_{j=1}^{n} H_{ij}(u_1, \ldots, u_n) u_j = 0 \qquad \text{in } \mathscr{D}, \quad i = 1, \ldots, n \tag{3.1}$$

$$u_i(x) = 0, \qquad x \in \delta\mathscr{D}, \quad i = 1, \ldots, n. \tag{3.2}$$

Define

$$h'_{ij} = \inf\{H_{ij}(u_1, \ldots, u_n) \mid u_k \geqslant 0, k = 1, \ldots, n\},$$

$$h''_{ij} = \sup\{H_{ij}(u_1, \ldots, u_n) \mid u_k \geqslant 0, k = 1, \ldots, n\},$$

$i, j = 1, \ldots, n$. The functions $H_{ij}(u_1, \ldots, u_n)$ are assumed to belong to the class C^α in the set $\{(u_1, \ldots, u_n) \mid u_k \geqslant 0, k = 1, \ldots, n\}$, i.e., they are locally Hölder continuous in $(u_1, \ldots, u_n)$ with Hölder exponent $\alpha, 0 < \alpha < 1$. Let $C^{2+\alpha}(\bar{\mathscr{D}})$ denote the Banach space of real-valued functions in $\bar{\mathscr{D}}$, with first and second derivatives also continuous in $\bar{\mathscr{D}}$, with finite value for the usual norm $|u|_{\mathscr{D}}^{(2+\alpha)}$. We assume the boundary $\delta\mathscr{D}$ belongs to class $C^{2+\alpha}$ (see, e.g., [5] for details of these symbols). The following conditions will be assumed:

(P1) $-\infty < h'_{ii} \leqslant h''_{ii} < \infty$, $i = 1, \ldots, n$;

$$0 < h'_{1j}, \qquad j = 2, \ldots, n;$$

$$0 \leqslant h'_{ij} \leqslant h''_{ij} < \infty, \qquad 1 \leqslant i, j \leqslant n, i \neq j,$$

for each $i = 2, \ldots, n$, at least one of $h'_{i1}, h'_{i2}, \ldots, h'_{ii-1}$ is positive.

(P2) In the set $M =^{(\mathrm{def})} \{(u_1, \ldots, u_n) \mid u_k \geqslant 0, k = 1, \ldots, n\}$,

$H_{11}(u_1, \ldots, u_n)$ is continuously differentiable with respect to $u_2, \ldots, u_n$;

and $|\partial H_{11}/\partial u_j| < K$ for all $(u_1, \ldots, u_n) \in M$, $j = 2, \ldots, n$, where K is some positive constant.

(P3) There exist positive constants p and U^* such that $H_{11}(u_1, \ldots, u_n) \leqslant -p$ for all $(u_1, \ldots, u_n) \in M$ with $u_1 \geqslant U^*$.

Note. (P1) is analogous to (C1), (C2) in Section 1. However, $0 < h'_{1j}$, $j = 2, \ldots, n$, is additional.

THEOREM 3.1. *Let $\hat{H}$ be an $n \times n$ matrix, $\hat{H} = (\hat{H}_{ij})$, $1 \leqslant i, j \leqslant n$, where $\hat{H}_{ij} \equiv^{\mathrm{def}} h''_{ij}$ except $(i, j) = (1, 1)$, and $\hat{H}_{11} \equiv^{\mathrm{def}} -p$. Suppose that* (P1)–(P3) *are satisfied. Further, let*

$$H_{11}(0, \ldots, 0) > \lambda_1 \tag{3.3}$$

and

$$\hat{H}\mathbf{c} \leqslant 0 \text{ for some positive } n \text{ vector } \mathbf{c} > 0. \tag{3.4}$$

Then the boundary value problem (3.1) *to* (3.2) *has a solution* $(\hat{u}_1(x), \ldots, \hat{u}_n(x))$ *with components in* $C^{2+\alpha}(\bar{\mathscr{D}})$ *and* $\hat{u}_i(x) > 0$ *in* $\mathscr{D}$, $i = 1, \ldots, n$. *($\hat{H}\mathbf{c} \leqslant 0$ means each of its component is $\leqslant 0$).*

Proof. We will construct upper and lower solutions for (3.1) to (3.2) and apply a theorem in [16] to conclude the existence of a positive solution. By (3.3), there is a small $\alpha > 0$ so that $H_{11}(u, 0, \ldots, 0) > \lambda_1$ for $0 \leqslant u < \alpha$. Choose $0 < \varepsilon < \min\{\alpha, K^{-1}h'_{12}, \ldots, K^{-1}h'_{1n}\}$. For each $i = 3, \ldots, n$, let $\hat{\imath}$ denote one of the j, $1 \leqslant j \leqslant i - 1$ so that $h'_{i\hat{\imath}} > 0$ (such $h'_{i\hat{\imath}}$ exist by (P1)). Choose $0 < \delta_2 < h'_{21} \cdot \varepsilon \cdot [|h'_{22} - \lambda_1|]^{-1}$, $\delta_1 = \varepsilon$, and $0 < \delta_i < h'_{i\hat{\imath}}\delta_{\hat{\imath}}[|h'_{ii} - \lambda_1|]^{-1}$ for $i = 3, \ldots, n$. (For $i = 2, \ldots, n$, if $h'_{ii} - \delta_1 = 0$, let $\delta_i > 0$ be arbitrary.) Define lower solutions as

$$v_1(x) = \varepsilon\omega(x), \qquad v_i(x) = \delta_i\omega(x), \quad i = 2, \ldots, n \tag{3.5}$$

for $x \in \mathscr{D}$. Define upper solutions as

$$w_i(x) \equiv c_i, \qquad i = 1, \ldots, n, \; x \in \bar{\mathscr{D}}, \tag{3.6}$$

where $\mathbf{c} = \mathrm{col}(c_1, \ldots, c_n)$ as stated in (3.4). (Note that without loss of generality, we may assume $c_1 \geqslant U^*$, and $c_i > \delta_i$, $i = 2, \ldots, n$.) We now check the appropriate inequalities for the v_i, w_i, $i = 1, \ldots, n$. We have

$$\begin{aligned} &\Delta v_1 + H_{11}(v_1, u_2, \ldots, u_n)v_1 + \sum_{j=2}^{n} H_{1j}(v_1, u_2, \ldots, u_n)u_j \\ &\quad \geqslant \varepsilon\omega(x)[-\lambda_1 + H_{11}(\varepsilon\omega(x), u_2, \ldots, u_n)] + \sum_{j=2}^{n} h'_{1j}u_j \end{aligned} \tag{3.7}$$

for $u_j \geqslant 0$, $j = 2, \ldots, n$, $x \in \mathscr{D}$. However, $[-\lambda_1 + H_{11}(\varepsilon\omega, 0, \ldots, 0)] > 0$ in $\mathscr{D}$; and $F(s, x, u_2, \ldots, u_n) \equiv^{(\mathrm{def})} \varepsilon\omega(x)[-\lambda_1 + H_{11}(\varepsilon\omega(x), \; su_2, \ldots, su_n)] +$

$\sum_{j=2}^{n} h'_{1j} s u_j$ is an increasing function of $s \geqslant 0$, for fixed $u_j \geqslant 0$, $j = 2,..., n$, each $x \in \mathscr{D}$ (by the choice of ε). Consequently, we have

$$\Delta v_1(x) + H_{11}(v_1(x), u_2,..., u_n)v_1 + \sum_{j=2}^{n} H_{1j}(v_1(x), u_2,..., u_n)u_j > 0 \quad (3.8)$$

for all $u_j \geqslant 0$, $j = 2,..., n$, $x \in \mathscr{D}$. For $i = 3,..., n$, we have

$$\begin{aligned} &\Delta v_i(x) + H_{ii}(u_1,..., u_{i-1}, v_i, u_{i+1},..., u_n)v_i \\ &\qquad + \sum_{\substack{j=1 \\ j \neq i}}^{n} H_{ij}(u_1,..., v_i,..., u_n)u_j \\ &\quad \geqslant \delta_i \omega(x)[-\lambda_1 + h'_{ii}] + h'_{ii} u_i \geqslant \omega(x)\{-\delta_i\, |h'_{ii} - \lambda_1| + h'_{ii}\, \delta_i\} \geqslant 0 \end{aligned} \quad (3.9)$$

for $v_j(x) \leqslant u_j \leqslant w_j(x)$, $j \neq i$, $x \in \mathscr{D}$ (by the choice of δ_i). For the case $i = 2$, all the inequalities in (3.9) are true, with h'_{ii}, u_i, and δ_i replaced, respectively, by h'_{21}, u_1, and ε. For the upper solutions, we have

$$\begin{aligned} &\Delta w_i(x) + H_{ii}(u_1,..., u_{i-1}, w_i, u_{i+1},..., u_n)\, w_i(x) \\ &\qquad + \sum_{\substack{j=1 \\ j \neq i}}^{n} H_{ij}(u_1,..., w_i,..., u_n)u_j \\ &\quad \leqslant \sum_{j=1}^{n} \hat{H}_{ij} c_j \leqslant 0 \end{aligned} \quad (3.10)$$

for each $i = 1,..., n$, $v_j(x) \leqslant u_j \leqslant w_j(x)$, $j \neq i$, $x \in \mathscr{D}$ (by the choice of c_j, (3.4) and (3.6)). By [16], (3.8)–(3.10) imply that there exists a solution $(\hat{u}_1(x),..., \hat{u}_n(x))$ as described in the statement of theorem with $v_i(x) \leqslant \hat{u}_i(x) \leqslant w_i(x)$, $i = 1,..., n$, $x \in \bar{\mathscr{D}}$. Consequently, $\hat{u}_i(x) > 0$ in $\mathscr{D}$, $i = 1,..., n$.

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